ON COEFFICIENT MEANS OF CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

BY

F. HOLLAND AND J. B. TWOMEY

ABSTRACT. Let $\mathcal{R}$ denote the class of regular functions whose derivatives have positive real part in the unit disc γ and let $\mathcal{S}$ denote the class of functions starlike in γ. In this paper we investigate the rates of growth of the means $s_n(\lambda) = n^{-1} \sum_1^n |a_k|^\lambda$ $(0 < \lambda \le 1)$ and $t_n(\lambda) = n^{-1} \sum_1^n k^\lambda |a_k|^\lambda$ $(\lambda > 0)$ as $n \to +\infty$ for bounded $f(z) = \sum_1^\infty a_k z^k \in \mathcal{R} \cup \mathcal{S}$. It is proved, for example, that the estimate $t_n(\lambda) = o(1)(\log n)^{-\alpha(\lambda)}$ $(n \to +\infty)$, where $\alpha(\lambda) = \lambda/2$ for $0 < \lambda < 2$ and $\alpha(\lambda) = 1$ for $\lambda \ge 2$, holds for such functions f, and that it is best possible for each fixed $\lambda > 0$ within the class $\mathcal{R}$ and for each fixed $\lambda \ge 2$ within the class $\mathcal{S}$.

It is also shown that the inequality $s_n(1) = o(1)n^{-1}(\log n)^{1/2}$, which holds for all bounded univalent functions, cannot be improved for bounded $f \in \mathcal{R}$. The behavior of $t_n(\lambda)$ as $n \to +\infty$ when $a_k \ge 0$ $(k \ge 1)$ and $\lambda \ge 1$ is also examined.

1. **Introduction and statement of results.** Let $\mathcal{R}$ denote the class of regular functions of the form

$$(1.1) \qquad f(z) = \sum_1^\infty a_n z^n \qquad (|z| < 1)$$

which satisfy $\operatorname{Re} f'(z) > 0$ in $\gamma = \{z: |z| < 1\}$, let $\mathcal{S}$ denote the class of regular functions (1.1) which are starlike in γ, and let $\mathcal{L}$ denote the class of functions f for which $f(z) = \log(g(z)/z)$ $(|z| < 1)$, where $g \in \mathcal{S}$. It is well known that if $f \in \mathcal{R} \cup \mathcal{S}$, then f is univalent in γ. It is also well known that $f \in \mathcal{S}$ iff $h \in \mathcal{R}$, where $h'(z) = zf'(z)/f(z)$ and $h(0) = 0$, and it follows easily that we also have $f \in \mathcal{L}$ iff $k \in \mathcal{R}$, where

$$(1.2) \qquad k(z) = \int_0^z [\zeta f'(\zeta) + 1] d\zeta = z + \sum_{n=1}^\infty \frac{n}{n+1} a_n z^{n+1}.$$

If $f \in \mathcal{R}$, then, by a classical result for functions with positive real part, $n|a_n| \le 2 \operatorname{Re} a_1$ $(n \ge 2)$. Applying this result to the function k defined in (1.2), we deduce that if $f \in \mathcal{L}$, then $n|a_n| \le 2$ $(n \ge 1)$. For $f \in \mathcal{S}$, on the other hand, we have [2] $n|a_n| = O(1)$ $(n \to +\infty)$ provided f is also bounded in γ. Combining these results we get

$$(1.3) \qquad |a_n| = O(1)/n \qquad (n \to +\infty)$$

for every bounded $f \in \mathcal{R} \cup \mathcal{S} \cup \mathcal{L}$. In this paper we investigate the growth of certain means of the coefficients $\{a_n\}$ for bounded $f \in \mathcal{R} \cup \mathcal{S} \cup \mathcal{L}$. We prove the following theorem.

Theorem 1. *Let $f(z) = \sum_1^\infty a_n z^n$ be regular and bounded in γ and suppose that $f \in \mathcal{R} \cup \mathcal{S} \cup \mathcal{L}$. Then, as $N \to +\infty$,*

$$(1.4) \qquad \sum_{n=1}^N |a_n|^\lambda = o(1)N^{1-\lambda}(\log N)^{-\lambda/2}, \qquad 0 < \lambda < 1,$$
$$= o(1)(\log N)^{1/2}, \qquad \lambda = 1,$$

and

$$(1.5) \qquad \sum_{n=1}^N n^\lambda |a_n|^\lambda = o(1)N(\log N)^{-\lambda/2}, \qquad 0 < \lambda < 2,$$
$$= o(1)N(\log N)^{-1}, \qquad \lambda \geq 2.$$

If, further, $a_n \geq 0$ for each $n \geq 1$, then, as $N \to +\infty$,

$$(1.6) \qquad \sum_{n=1}^N{}' n^\lambda a_n^\lambda = o(1)N(\log N)^{-1}, \qquad \lambda \geq 1.$$

Inequalities (1.4), (1.5) and (1.6) are best possible within each of the classes $\mathcal{R}$ and $\mathcal{L}$, while (1.6), and hence (1.5) for $\lambda \geq 2$, is best possible within the class $\mathcal{S}$.

It has already been noted in [4] that inequality (1.4), with $\lambda = 1$, is valid for every f in $\mathcal{B}$, the class of functions univalent and bounded in γ, and this of course implies the inequality for bounded $f \in \mathcal{R} \cup \mathcal{S}$. Also, the examples discussed in [4] can be used to show that the inequality is best possible within the class $\mathcal{B}$, but this does not imply the stronger result that (1.4) is best possible for bounded $f \in \mathcal{R}$. Whether (1.4) can be improved for the class of bounded starlike functions remains an open question.

In view of (1.6), it seems worth remarking that the functions we construct to show that (1.4) and (1.5) are best possible have real coefficients.

2. Proof of Theorem 1. The positive results. We first prove (1.5). It was shown in [6] that, for bounded $f \in \mathcal{R} \cup \mathcal{S}$,

$$(2.1) \qquad 2\pi \sum_{n=1}^\infty n^2 |a_n|^2 r^{2n} = r^2 \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta$$
$$= o(1)[(1 - r)\log(1/(1 - r))]^{-1},$$

as $r \to 1$. If $f \in \mathcal{L}$ is bounded in γ and $k \in \mathcal{R}$ is defined as in (1.2), then, because of (1.3), k is also bounded in γ. Applying (2.1) to k, we obtain immediately that (2.1) also holds for bounded $f \in \mathcal{L}$. It follows easily from (2.1) that (1.5) holds for $\lambda = 2$ and hence also by (1.3) for $\lambda > 2$. If $0 < \lambda < 2$, then by Hölder's inequality

$$\sum_{n=1}^{N} n^{\lambda} |a_n|^{\lambda} \leq \left(\sum_{1}^{N} n^2 |a_n|^2 \right)^{\lambda/2} N^{(2-\lambda)/2}$$

$$= o(1) N (\log N)^{-\lambda/2},$$

as $N \to +\infty$, where we have used (1.5) with $\lambda = 2$. This completes the proof of (1.5).

We now prove (1.4). Write $T_n(\lambda) = \sum_{k=1}^{n} k^{\lambda} |a_k|^{\lambda}$ for $\lambda > 0$ and $n \geq 1$. Then by partial summation, for $N \geq 2$,

(2.2)
$$\sum_{n=1}^{N} |a_n|^{\lambda} = \sum_{n=1}^{N-1} \{ n^{-\lambda} - (n+1)^{-\lambda} \} T_n(\lambda) + N^{-\lambda} T_N(\lambda)$$

$$\leq \lambda \sum_{n=1}^{N-1} n^{-\lambda-1} T_n(\lambda) + N^{-\lambda} T_N(\lambda).$$

By (1.5), we thus have, as $N \to +\infty$,

$$\sum_{n=1}^{N} |a_n|^{\lambda} = o(1) \sum_{n=2}^{N} n^{-\lambda} (\log n)^{-\lambda/2} + o(1) N^{1-\lambda} (\log N)^{-\lambda/2}$$

$$= o(1) N^{1-\lambda} (\log N)^{-\lambda/2},$$

for $0 < \lambda < 1$, and

$$\sum_{n=1}^{N} |a_n| = o(1) \sum_{n=2}^{N} n^{-1} (\log n)^{-1/2} + o(1) (\log N)^{-1/2}$$

$$= o(1) (\log N)^{1/2}.$$

This proves (1.4).

To prove (1.6) we note that it has been shown in [7] that, for bounded $f \in \mathcal{S}$,

(2.3)
$$f'(r) = \sum_{n=1}^{\infty} n a_n r^{n-1} = o(1) \left[(1-r) \log \frac{1}{1-r} \right]^{-1},$$

as $r \to 1$, and that it is not too difficult to deduce that (2.3) holds for bounded $f \in \mathcal{R} \cup \mathcal{L}$ as well. If $a_n \geq 0$ for $n \geq 1$, then it is an easy consequence of (2.3) that (1.6) holds for $\lambda = 1$ and hence also by (1.3) for $\lambda > 1$. This proves (1.6).

3. The counter examples. To prove the negative assertions of Theorem 1, we need the following theorem.

Theorem 2. *Let* $\{ \varepsilon(n) \}_1^{\infty}$ *be a sequence of positive terms which decrease to 0 as* $n \uparrow +\infty$. *Let* $0 < \lambda \leq 1$. *Then there exist regular functions*

$$f_{\lambda}(z) = 1 + \sum_{n=1}^{\infty} c_n(\lambda) z^n,$$

where $c_n(\lambda)$ *is real for each n, for which* $\operatorname{Re} f_{\lambda}(z) > 0$ *in* γ,

(3.1)
$$\sup_{z \in \gamma} \left| \int_0^z \frac{f_{\lambda}(\zeta) - 1}{\zeta} d\zeta \right| < +\infty,$$

and, as $N \to +\infty$,

(3.2)
$$S(N,\lambda) \neq O(1)\varepsilon(N)N^{1-\lambda}(\log N)^{-\lambda/2}, \quad 0 < \lambda < 1,$$
$$S(N,1) \neq O(1)\varepsilon(N)(\log N)^{1/2},$$

where $S(N,\lambda) = \sum_{n=1}^{N} n^{-\lambda}|c_n(\lambda)|^\lambda$ for $0 < \lambda \leq 1$ and $N \geq 1$.

Proof of Theorem 2. We begin be defining a convex increasing sequence $\{n_k\}_0^\infty$ of nonnegative even integers for which

(3.3)
$$k^5 < n_k \leq (k+1)^5, \quad k \geq 1,$$

and

(3.4) *the equation $n = n_k \pm n_j$ $(k \geq j \geq 1)$ has at most one solution for each positive integer n.*

Following Marcinkiewicz [5, p. 43], we define the sequence inductively as follows:

Let $n_0 = 0$, $n_1 = 2$ and suppose that, for $k \geq 2$, $n_0, n_1, \ldots, n_{k-1}$ have been defined. Then there are at most $4k^3$ positive integers of the form $\pm n_j \pm n_l \pm n_p$ where $0 \leq j, l, p \leq k - 1$. Hence there is (at least) one even integer, denote it by n_k, satisfying

$$(k+1)^5 - (8k^3 + 2) \leq n_k \leq (k+1)^5$$

which is not of the form $\pm n_j \pm n_l \pm n_p$ $(0 \leq j, l, p \leq k - 1)$. It is easily verified that the increasing sequence $\{n_k\}$ thus defined is convex and satisfies (3.3) and (3.4).

For $|z| < 1$ and $m \geq 1$, we now set

(3.5)
$$F_m(z) = \frac{1}{2(m+1)} \sum_{k=0}^{m} \left\{ \frac{1 + ze^{i\theta_k}}{1 - ze^{i\theta_k}} + \frac{1 + ze^{-i\theta_k}}{1 - ze^{-i\theta_k}} \right\},$$

where $\theta_0 = 0$ and $\theta_k \equiv \theta_k(m) = \pi n_k/2n_m$ for $1 \leq k \leq m$.

4. To complete the proof of Theorem 2 we need two lemmas.

Lemma 1. *Let $F_m(z)$ be defined as in (3.5). Then there is an absolute constant M such that*

(4.1)
$$\left| \int_0^z \frac{F_m(\zeta) - 1}{\zeta} d\zeta \right| \leq M$$

for $|z| \leq 1 - e^{-m}$ and $m \geq 1$.

Proof. If $G_m(z) \equiv \int_0^z ((F_m(\zeta) - 1)/\zeta) d\zeta$, then, by (3.5),

(4.2)
$$G_m(z) = \frac{1}{m+1} \sum_{k=0}^{m} \{l(z, \theta_k) + l(z, -\theta_k)\}$$

where $l(z, \psi) = -\log(1 - ze^{i\psi})$. Since

$$|\text{Im } l(z, \psi)| \le \pi/2$$

and

$$\text{Re } l(z, \psi) \ge \log(1/(1 + r)) \ge -\log 2$$

for $|z| = r < 1$ and $-\pi \le \psi \le \pi$, it follows immediately that if $G_m(z) = U_m(z) + iV_m(z)$, then

$$|V_m(z)| \le \pi \quad \text{and} \quad U_m(z) \ge -2\log 2$$

for $|z| < 1$ and $m \ge 1$. To complete the proof of (4.1), it is thus sufficient to show that, for $|z| \le 1 - e^{-m}$ and $m \ge 1$,

(4.3) $$U_m(z) < A,$$

where A is some absolute constant.

Write $u(r, \psi) = -\log|1 - re^{i\psi}|$. Then, by (4.2),

(4.4) $$U_m(z) = \frac{1}{m+1} \sum_{k=0}^{m} \{u(r, \theta + \theta_k) + u(r, \theta - \theta_k)\},$$

where $z = re^{i\theta}$. We prove first that

(4.5) $$\sum_{k=0}^{m} u(r, \theta - \theta_k) \le 2 \sum_{k=0}^{m} u(r, \theta_k)$$

for $0 \le r < 1$ and $-\pi \le \theta \le \pi$. Suppose, to begin with, that $0 \le \theta \le \pi/2$, so that there is an integer j $(1 \le j \le m)$ such that $\theta_{j-1} \le \theta \le \theta_j$. For fixed r, $u(r, \psi)$ is a decreasing function of ψ in $[0, \pi]$, and so

$$\sum_{k=0}^{m} u(r, \theta - \theta_k) \le \sum_{k=0}^{j-1} u(r, \theta_{j-1} - \theta_k) + \sum_{k=j}^{m} u(r, \theta_k - \theta_j)$$

$$\le \sum_{k=0}^{j-1} u(r, \theta_{j-k-1}) + \sum_{k=j}^{m} u(r, \theta_{k-j})$$

$$\le 2 \sum_{k=0}^{m} u(r, \theta_k),$$

since the convexity of the sequence $\{\theta_k\}$ implies that $\theta_n - \theta_p \ge \theta_{n-p} - \theta_0 = \theta_{n-p}$ for all p and n for which $p \le n \le m$. This proves (4.5) for $0 \le \theta \le \pi/2$. It is easy to see that the above argument can be modified to establish (4.5) for the remaining values of θ in $[-\pi, \pi]$ and we have thus proved (4.5).

It follows from (4.4) and (4.5), and the fact that $u(r, \psi)$ is an even function of ψ, that

(4.6) $$U_m(z) \le \frac{4}{m+1} \sum_{k=0}^{m} u(r, \theta_k)$$

for $|z| < 1$. But

$$|1 - re^{i\theta_k}|^2 \geq 4r\sin^2\tfrac{1}{2}\theta_k \geq (4r/\pi^2)\theta_k^2 = rn_k^2/n_m^2,$$

and so, if $r = 1 - e^{-m}$,

$$\sum_{k=0}^{m} u(r,\theta_k) \leq u(r,0) + \sum_{k=1}^{m} \log\frac{n_m}{r^{1/2}n_k}$$

$$\leq \log\frac{1}{1-r} + \sum_{k=1}^{m}\log\frac{(m+1)^5}{k^5} + \frac{m}{2}\log\frac{1}{r}$$

$$\leq m + 5m\log(m+1) - 5[m\log m - m] + (m/2)\log 2$$

$$= m[6 + 5\log((m+1)/m) + \tfrac{1}{2}\log 2],$$

where we have used (3.3) and the inequality $\sum_{k=1}^{m}\log k \geq m\log m - m$, ($m \geq 1$). By (4.6) we thus have, for $|z| = 1 - e^{-m}$,

$$U_m(z) \leq (4m/(m+1))[6 + (11/2)\log 2],$$

and (4.3) follows immediately by the maximum principle for harmonic functions. This proves Lemma 1.

Lemma 2. Let $F_m(z) = 1 + \sum_{1}^{\infty} b_n(m)z^n$ for $|z| < 1$ and $m \geq 1$. Then if $N = 2n_m$ and $0 < \lambda \leq 2$, we have

$$(4.7) \qquad\qquad \frac{1}{N}\sum_{n=1}^{N}|b_n(m)|^\lambda \geq \frac{A(\lambda)}{m^{\lambda/2}},$$

where $A(\lambda)$ is a positive constant which depends only on λ.

Proof (cf. proof of Szidon's Lemma [1, p. 258]). By (3.5) we have

$$b_n(m) = \frac{2}{m+1}\sum_{k=0}^{m}\cos n\theta_k.$$

Hence, writing b_n for $b_n(m)$ and using the fact that

$$\cos n\theta_k\cos n\theta_j = \tfrac{1}{2}\{\cos n(\theta_k + \theta_j) + \cos n(\theta_k - \theta_j)\},$$

we obtain

$$(4.8) \qquad (m+1)^2 b_n^2 = 2\sum_{j,k=0}^{m}\left\{\cos\frac{n\pi}{N}(n_k + n_j) + \cos\frac{n\pi}{N}(n_k - n_j)\right\}$$

$$= 2\sum_{s=0}^{N}\alpha_s\cos\frac{n\pi s}{N},$$

say. We note that $\alpha_0 = m + 2$ and that $\alpha_s = 0$ when s is odd. It follows easily that

$$(4.9) \qquad \frac{1}{N} \sum_{n=1}^{N} b_n^2 = \frac{2^{a_0}}{(m+1)^2} = \frac{2(m+2)}{(m+1)^2} \geq \frac{1}{m},$$

since

$$(4.10) \qquad \begin{aligned} \sum_{n=1}^{N} \cos \frac{\pi x n}{N} &= \mathrm{Re}\left\{ \sum_{n=1}^{N} e^{\pi i x n / N} \right\} \\ &= \mathrm{Re}\left\{ e^{\pi i x / N} \frac{1 - e^{\pi i x}}{1 - e^{\pi i x / N}} \right\} = 0, \end{aligned}$$

when x is an even integer such that $x \not\equiv 0 \pmod{2N}$.

We note that, by (4.8),

$$(m+1)^4 b_n^4 = 4 \sum_{s,t=0}^{N} \alpha_s \alpha_t \left\{ \cos \frac{n\pi}{N}(s+t) + \cos \frac{n\pi}{N}(s-t) \right\}$$

and so, using (4.10) again and the fact that N is even, we obtain

$$\frac{1}{N} \sum_{n=1}^{N} b_n^4 = \frac{4}{(m+1)^4} \left\{ \alpha_0^2 + \alpha_N^2 + \sum_{s=0}^{N} \alpha_s^2 \right\}.$$

But, for $s \geq 1$, we have, because of (3.4) and (4.8), $\alpha_s \leq 4$ whenever $s = n_k + n_j$ or $s = |n_k - n_j|$ for any j and k, and $\alpha_s = 0$ otherwise. Hence

$$(4.11) \quad \frac{1}{N} \sum_{n=1}^{N} b_n^4 \leq \frac{4}{(m+1)^4} \{2(m+2)^2 + 16 + 16[2(m+1)^2]\} \leq \frac{B}{m^2},$$

where B is an absolute constant.

Finally, by Hölder's inequality, for $0 < \lambda < 2$,

$$\frac{1}{N} \sum_{n=1}^{N} |b_n|^2 \leq \left(\frac{1}{N} \sum_{1}^{N} |b_n| \right)^{2/(4-\lambda)} \left(\frac{1}{N} \sum_{1}^{N} |b_n|^4 \right)^{(2-\lambda)/(4-\lambda)}$$

which, by (4.9) and (4.11), implies that

$$\frac{1}{N} \sum_{n=1}^{N} |b_n|^\lambda \geq A(\lambda) \left\{ \frac{m^{2(2-\lambda)/(4-\lambda)}}{m} \right\}^{(4-\lambda)/2} = \frac{A(\lambda)}{m^{\lambda/2}},$$

as required. This completes the proof of Lemma 2.

5. We now complete the proof of Theorem 2.

Let λ be fixed with $0 < \lambda \leq 1$. Choose an increasing sequence $\{m_k\}_1^\infty$ of positive integers for which

$$(5.1) \qquad \varepsilon(m_{k+1})^{1/2\lambda} < \min\{\tfrac{1}{2}\varepsilon(m_k)^{1/2\lambda}, m_k^{-1}\}$$

and

$$(5.2) \qquad (1 - r_k^\lambda)^{-1} < m_{k+1},$$

for $k \geq 1$, where $r_k = 1 - e^{-m_k}$ $(k \geq 1)$. We now define

$$(5.3) \qquad f_\lambda(z) = \sum_{k=1}^{\infty} \varepsilon_k F_{m_k}(r_k z)$$

for $|z| < 1$, where $\varepsilon_k = \varepsilon(m_k)^{1/2\lambda}$ $(k \geq 1)$ and $F_m(z)$ is defined by (3.5). By (5.1), $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$ $(k \geq 1)$ and it follows easily that $\sum_1^\infty \varepsilon_k < +\infty$. We assume, as we may, that

$$(5.4) \qquad \sum_{k=1}^{\infty} \varepsilon_k = 1.$$

It is clear from (5.3) and (3.5) that $\text{Re } f_\lambda(z) > 0$ in γ, and it follows easily from (5.3), (5.4) and (4.1) that (3.1) also holds.

If $f_\lambda(z) = 1 + \sum_1^\infty c_n(\lambda) z^n$, then

$$(5.5) \qquad S(N,\lambda) \equiv \sum_{n=1}^{N} n^{-\lambda} |c_n(\lambda)|^\lambda = \sum_{n=1}^{N} n^{-\lambda} \left| \sum_{k=1}^{\infty} \varepsilon_k r_k^n b_n(m_k) \right|^\lambda,$$

where $\{b_n(m)\}$ is defined as in Lemma 2. Since $|a + b|^\lambda \leq 2^\lambda(|a|^\lambda + |b|^\lambda)$, we have, for $t \geq 2$ and $n \geq 1$,

$$\left| \sum_{k=1}^{\infty} \varepsilon_k r_k^n b_n(m_k) \right|^\lambda \geq 2^{-\lambda}[\varepsilon_t r_t^n |b_n(m_t)|]^\lambda - \left| \left\{ \sum_{k=1}^{t-1} + \sum_{k=t+1}^{\infty} \right\} \varepsilon_k r_k^n |b_n(m_k)| \right|^\lambda$$

$$\geq 2^{-\lambda}[\varepsilon_t r_t^n |b_n(m_t)|]^\lambda - 2^\lambda \left[2 r_{t-1}^n \sum_{k=1}^{t-1} \varepsilon_k \right]^\lambda - 2^\lambda \left[2 \sum_{k=t+1}^{\infty} \varepsilon_k \right]^\lambda$$

since $|b_n(m_k)| \leq 2$ for $n \geq 1$ and $k \geq 1$. But, for $t \geq 2$, $\sum_{k=1}^{t-1} \varepsilon_k \leq 1$, by (5.6), and, by (5.1),

$$\sum_{k=t+1}^{\infty} \varepsilon_k \leq \varepsilon_{t+1} \sum_{k=1}^{\infty} 2^{-k+1} = 2\varepsilon_{t+1} \leq 2 m_t^{-1},$$

so that

$$(5.6) \qquad \left| \sum_{k=1}^{\infty} \varepsilon_k r_k^n b_n(m_k) \right|^\lambda \geq 2^{-\lambda}[\varepsilon_t r_t^n |b_n(m_t)|]^\lambda - 4^\lambda r_{t-1}^{n\lambda} - 8^\lambda m_t^{-\lambda}.$$

Write now, for $t \geq 1$, $N_t = 2 n_{m_t}$, where $\{n_k\}$ is the sequence defined in §3, write $p_t = [N_t^{-1} e^{m_t}]$ and $P_t = p_t N_t$. Then, by (5.5) and (5.6), for $t \geq 2$,

$$(5.7) \quad S(P_t,\lambda) \geq 2^{-\lambda} \varepsilon_t^\lambda r_t^{\lambda P_t} \sum_{n=1}^{P_t} n^{-\lambda} |b_n(m_t)|^\lambda - 4^\lambda \sum_{n=1}^{\infty} n^{-\lambda} r_{t-1}^{n\lambda} - 8^\lambda m_t^{-\lambda} \sum_{n=1}^{P_t} n^{-\lambda}.$$

We note that, as $t \to +\infty$, we have

$$\sum_{n=1}^{\infty} n^{-\lambda} r_{t-1}^{n\lambda} = O(1)(1 - r_{t-1}^{\lambda})^{\lambda-1}, \qquad 0 < \lambda < 1,$$

(5.8)
$$= O(1)\log(1 - r_{t-1})^{-1}, \qquad \lambda = 1,$$

$$= O(1)m_t^{1-\lambda}, \qquad 0 < \lambda < 1,$$

$$= O(1)\log m_t, \qquad \lambda = 1,$$

by (5.2), and

(5.9)
$$\sum_{n=1}^{P_t} n^{-\lambda} = O(1)P_t^{1-\lambda}, \qquad 0 < \lambda < 1,$$

$$= O(1)\log P_t, \qquad \lambda = 1.$$

We note too that, for $0 < \lambda \le 1$ and $t \ge 1$,

(5.10)
$$r_t^{\lambda P_t} \ge \exp[-\lambda e^{m_t} \log r_t^{-1}]$$

$$= \exp[-\lambda \log(1 - r_t)^{-1} \log r_t^{-1}] \ge \exp[-\lambda].$$

Also, $b_n(m_t)$ is a periodic function of n of period N_t, and so

$$\sum_{n=1}^{P_t} n^{-\lambda} |b_n(m_t)|^{\lambda} = \sum_{k=0}^{p_t-1} \sum_{s=1}^{N_t} \frac{|b_s(m_t)|^{\lambda}}{[kN_t + s]^{\lambda}}$$

$$\ge \sum_{k=0}^{p_t-1} [(k + 1)N_t]^{-\lambda} \sum_{s=1}^{N_t} |b_s(m_t)|^{\lambda}$$

$$\ge A(\lambda) N_t^{1-\lambda} m_t^{-\lambda/2} \sum_{k=1}^{P_t} k^{-\lambda},$$

by (4.7). Hence, for $t \ge 1$,

(5.11)
$$\sum_{n=1}^{P_t} n^{-\lambda} |b_n(m_t)|^{\lambda} \ge A_1(\lambda) P_t^{1-\lambda} m_t^{-\lambda/2} \qquad 0 < \lambda < 1,$$

$$\ge B_1 m_t^{-1/2} \log p_t, \qquad \lambda = 1.$$

Here, and in (5.12) below, $A_i(\lambda)$ ($i = 1, 2, 3$) denote positive constants which depend only on λ, and B_i ($i = 1, 2, 3$) denote positive absolute constants. By (5.7), (5.8), (5.9), (5.10), (5.11), and the fact that $m_t = \log P_t + O(1)$ and $\log p_t = (1 + o(1))\log P_t$ as $t \to +\infty$, we have

$$S(P_t, \lambda) \ge A_2(\lambda)\varepsilon(m_t)^{1/2} P_t^{1-\lambda}(\log P_t)^{-\lambda/2} - A_3(\lambda)P_t^{1-\lambda}(\log P_t)^{-\lambda},$$

(5.12)
$$0 < \lambda < 1,$$

$$\ge B_2 \varepsilon(m_t)^{1/2}(\log P_t)^{1/2} - B_3 \log\log P_t, \qquad \lambda = 1,$$

for $t \ge 2$. Since we may assume without loss of generality that $\{\varepsilon(n)\}$ does not decrease to 0 too rapidly, it is clear that (5.12) implies (3.2). This completes the proof of Theorem 2.

6. We now show that Theorem 2 implies that (1.4), for $0 < \lambda \le 1$, and (1.5), for $0 < \lambda \le 2$, are both best possible within each of the classes $\mathcal{R}$ and $\mathcal{L}$.

Let $\{\varepsilon(n)\}$ denote a sequence of positive terms which decrease to 0 as $n \uparrow \infty$ and let $0 < \lambda \le 1$. Let f_λ be defined as in the proof of Theorem 2 and set $g_\lambda(z) = \int_0^z f_\lambda(\zeta)\,d\zeta$ for $z \in \gamma$. Clearly $g_\lambda \in \mathcal{R}$. We show next that g_λ is bounded in γ. By integration by parts,

$$(6.1) \qquad g_\lambda(z) = z\left\{1 + \int_0^z \frac{f_\lambda(\zeta) - 1}{\zeta}\,d\zeta - \int_0^z \frac{g_\lambda(\zeta) - \zeta}{\zeta^2}\,d\zeta\right\}.$$

The first integral on the right of (6.1) is bounded, by (3.1), and the boundedness of the second integral follows easily from the fact that, for $z = re^{i\theta} \in \gamma$,

$$|g_\lambda(z)| \le \int_0^r |f_\lambda(\rho e^{i\theta})|\,d\rho \le \int_0^r \frac{1+\rho}{1-\rho}\,d\rho < 2\log\frac{1}{1-r},$$

where we have used a classical inequality for functions with positive real part. Hence g_λ is bounded in γ. Now, if $f_\lambda(z) = 1 + \sum_1^\infty c_n(\lambda)z^n$ and $g_\lambda(z) = \sum_1^\infty a_n(\lambda)z^n$ $(z \in \gamma)$, then, for $n \ge 1$,

$$|a_{n+1}(\lambda)| = |c_n(\lambda)/(n+1)| \ge \tfrac{1}{2}|c_n(\lambda)/n|.$$

Hence, by (3.2), as $N \to +\infty$,

$$\sum_{n=1}^N |a_n(\lambda)|^\lambda \ne O(1)\varepsilon(N)N^{\lambda-1}(\log N)^{-\lambda/2}, \quad \text{for } 0 < \lambda < 1,$$

and

$$\sum_{n=1}^N |a_n(1)| \ne o(1)\varepsilon(N)(\log N)^{1/2}.$$

This proves that (1.4) is best possible within the class $\mathcal{R}$.

To establish the same result for the class $\mathcal{L}$ we set

$$h_\lambda(z) = \int_0^z \frac{f_\lambda(\zeta) - 1}{\zeta}\,d\zeta \qquad (z \in \gamma, 0 < \lambda \le 1),$$

where f_λ is defined as above. Then, because of (3.1), h_λ is bounded in γ and, also, it is easily verified that $h_\lambda \in \mathcal{L}$. Since $h_\lambda(z) = \sum_1^\infty (c_n(\lambda)/n)z^n$ $(z \in \gamma)$, it follows immediately from (3.2) that (1.4) is best possible for the class $\mathcal{L}$ also.

It is an easy consequence of (2.2), and the fact that (1.4) is best possible for $0 < \lambda \le 1$, that (1.5) is also best possible within each of the classes $\mathcal{R}$ and $\mathcal{L}$ for $0 < \lambda \le 1$, and hence for $1 < \lambda \le 2$ as well; since

$$\frac{1}{N}\sum_{n=1}^N n|a_n| \le \left(\frac{1}{N}\sum_{n=1}^N n^\lambda |a_n|^\lambda\right)^{1/\lambda}$$

for $\lambda > 1$ [3, p. 26].

7. It remains to prove that (1.6) is best possible within each of the classes $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{L}$ for each fixed $\lambda \geq 1$. To do this we need the following theorem.

   Theorem 3. *Let* $\{\varepsilon(n)\}$ *denote a sequence of positive terms which decrease to* 0 *as* $n \uparrow \infty$. *Let* $\lambda \geq 1$. *Then there exists a regular function* $g_\lambda(z) = 1 + \sum_1^\infty b_n(\lambda)z^n$, *where* $b_n(\lambda)$ *is real and nonnegative for each* n, *for which* Re $g_\lambda(z) > 0$ $(z \in \gamma)$,

$$(7.1) \qquad \sup_{z \in \gamma} \left| \int_0^z \frac{g_\lambda(\zeta) - 1}{\zeta} d\zeta \right| < +\infty,$$

and, as $N \to +\infty$,

$$(7.2) \qquad \sum_{n=1}^N b_n(\lambda)^\lambda \neq O(1)\varepsilon(N)N(\log N)^{-1}.$$

   Proof. Let λ be fixed with $\lambda \geq 1$. Choose an increasing sequence $\{n_k\}_1^\infty$, of positive integers for which $n_1 = 2$,

$$(7.3) \qquad \sum_{k=1}^\infty \varepsilon(n_k)^{1/2\lambda} < +\infty,$$

and

$$(7.4) \qquad n_k \log n_k < \log n_{k+1}, \quad \text{for } k \geq 1.$$

For $n \geq 1$, write $G_n(z) = 1 + 2\sum_{k=1}^n (1 - k/(n + 1))z^k$. It is well known that, for $n \geq 1$, the Fejér kernel $\frac{1}{2}$ Re $G_n(e^{i\theta})$ is nonnegative for $\theta \in [-\pi, \pi]$, and it follows immediately that

$$(7.5) \qquad \text{Re } G_n(z) > 0 \qquad (z \in \gamma, n \geq 1).$$

For $z \in \gamma$ we now define

$$(7.6) \qquad g_\lambda(z) = \sum_{k=1}^\infty \varepsilon_k G_{n_k}(z^{m_k})$$

where $\varepsilon_k = \varepsilon(n_k)^{1/2\lambda}$ and

$$(7.7) \qquad m_k = \log n_k$$

for $k \geq 1$. We assume, as we may, that $\sum_1^\infty \varepsilon_k = 1$. Because of (7.4) and (7.7), the power series expansion of $g_\lambda(z)$ is obtained by simply removing brackets, that is

$$g_\lambda(z) = 1 + \sum_{n=1}^\infty b_n(\lambda)z^n$$
$$= 1 + 2\varepsilon_1 \sum_{s=1}^{n_1} \left(1 - \frac{s}{n_1 + 1}\right)z^{sm_1} + 2\varepsilon_2 \sum_{s=1}^{n_2} \left(1 - \frac{s}{n_2 + 1}\right)z^{sm_2} + \cdots.$$

Hence, for $z \in \gamma$,

$$\left| \int_0^z \frac{g_\lambda(\zeta) - 1}{\zeta} d\zeta \right| \leq \sum_{n=1}^\infty \frac{b_n(\lambda)}{n}$$

$$= 2 \sum_{k=1}^\infty \frac{\varepsilon_k}{m_k} \sum_{s=1}^{n_k} \left(1 - \frac{s}{n_k + 1} \right) \Big/ s$$

$$< 2 \sum_{k=1}^\infty \varepsilon_k \frac{1 + \log n_k}{m_k} < +\infty$$

by (7.3) and (7.7). This establishes (7.1).

For $k \geq 1$, write $N_k = n_k m_k$. Then, for $k \geq 1$,

$$\sum_{s=1}^{N_k} b_n(\lambda)^\lambda \geq 2^\lambda \varepsilon_k^\lambda \sum_{s=1}^{n_k} \left(1 - \frac{s}{n_k + 1} \right)^\lambda$$

$$= \left(\frac{2\varepsilon_k}{n_k + 1} \right)^\lambda \sum_{s=1}^{n_k} s^\lambda \geq \left(\frac{2\varepsilon_k}{n_k + 1} \right)^\lambda \frac{n_k^{\lambda+1}}{\lambda + 1}$$

$$> \frac{\varepsilon_k}{\lambda + 1} n_k = \frac{\varepsilon(n_k)^{1/2\lambda}}{\lambda + 1} N_k (\log n_k)^{-1}$$

$$> \frac{\varepsilon(n_k)^{1/2\lambda}}{\lambda + 1} N_k (\log N_k)^{-1},$$

and (7.2) follows immediately. This completes the proof of Theorem 3.

8. It is easily established, by arguments similar to those used in §6, that Theorem 3 implies that (1.6) is best possible within each of the classes $\mathscr{R}$ and $\mathscr{L}$.

To prove the same result for the class $\mathscr{S}$ we define, for $z \in \gamma$ and $\lambda \geq 1$,

$$(8.1) \qquad f_\lambda(z) = z \exp \int_0^z \frac{g_\lambda(\zeta) - 1}{\zeta} d\zeta,$$

where g_λ is defined as in the proof of Theorem 3. Then $f_\lambda \in \mathscr{S}$, f_λ is bounded in γ, and $f_\lambda(z) = \sum_1^\infty a_n(\lambda) z^n$ ($z \in \gamma$), where $a_n(\lambda) \geq 0$ for $n \geq 1$ and $\sum a_n(\lambda) < +\infty$. Since $f_\lambda(z)/z \neq 0$ for z in the closure of γ, it follows from Wiener's theorem [1, p. 194] that the power series expansion of $z/f_\lambda(z)$ is also absolutely convergent for $z = 1$, i.e.

$$\frac{z}{f_\lambda(z)} = \sum_0^\infty c_n(\lambda) z^n \qquad (z \in \gamma),$$

say, where $\sum |c_n(\lambda)| < +\infty$. Now, by (8.1),

$$g_\lambda(z) = \frac{z}{f_\lambda(z)} f_\lambda'(z) \qquad (z \in \gamma),$$

and so, for $n \geq 1$,

$$(8.2) \qquad b_n = \sum_{k=1}^{n+1} c_{n-k+1} k a_k$$

where for brevity we have suppressed λ. We show now that, for $\lambda \geq 1$ and $N \geq 1$,

$$(8.3) \qquad \sum_{n=1}^{N} b_n^\lambda \leq C^\lambda \sum_{n=1}^{N+1} n^\lambda a_n^\lambda$$

where $C = \sum_0^\infty |c_n| > 0$. We note first that, for $\lambda = 1$, (8.3) is an easy consequence of (8.2). Suppose next that $\lambda > 1$. Then, by (8.2) and Hölder's inequality,

$$(8.4) \qquad \begin{aligned} b_n^\lambda &\leq \sum_{k=1}^{n+1} |c_{n-k+1}| k^\lambda a_k^\lambda \left(\sum_{k=0}^{n} |c_k| \right)^{\lambda/\lambda'} \\ &\leq C^{\lambda/\lambda'} \sum_{k=1}^{n+1} |c_{n-k+1}| k^\lambda a_k^\lambda, \end{aligned}$$

where $\lambda' = \lambda/(\lambda - 1)$. But

$$\begin{aligned} \sum_{n=1}^{N} \sum_{k=1}^{n+1} |c_{n-k+1}| k^\lambda a_k^\lambda &\leq \sum_{k=0}^{N} |c_k| \sum_{k=1}^{N+1} k^\lambda a_k^\lambda, \\ &\leq C \sum_{k=1}^{N+1} k^\lambda a_k^\lambda, \end{aligned}$$

and this, combined with (8.4), clearly implies (8.3). It is obvious that (8.3) and (7.2) together imply that (1.6), and hence also (1.5) for $\lambda \geq 2$, is best possible within the class $\mathscr{S}$. We have thus completed the proof of Theorem 1.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, CORK, IRELAND